Section 6.1 — Linear Combinations of Vectors

The expression \( au + bv \), where \( a \) and \( b \) are scalar quantities, is a linear combination of the vectors \( u \) and \( v \). A linear combination of three vectors \( u, v, \) and \( w \) is written \( au + bv + cw \). These expressions are linear because they consist only of the sum of scalar multiples of vectors and nothing else.

In the preceding chapters, it was by means of such expressions that sets of vectors were combined to form new vectors. Recall, that one way to write an algebraic vector was as a linear combination of the basis vectors \( \hat{i}, \hat{j}, \) and \( \hat{k} \):
\[
a\hat{i} + b\hat{j} + c\hat{k}.
\]

This section is concerned with the reverse problem. Under what circumstances can a given vector be expressed as a linear combination of other vectors? That is, when can a vector \( \vec{x} \) be expressed in the form \( \vec{x} = au + bv + \ldots \)? How are the coefficients of such an expression to be found? What does it mean if forming such a linear combination proves to be impossible?

We begin the investigation of these questions with the simplest case. The vector \( au \) is a linear combination of the single vector \( u \). Under what circumstances is it possible to express a given vector \( \vec{x} \) in the form \( \vec{x} = au \)? The answer is that we write \( \vec{x} = au \), and determine a numerical value for the scalar \( a \) only when \( \vec{x} \) and \( u \) are collinear. The equation \( \vec{x} = au \) implies that the two vectors \( \vec{x} \) and \( u \) are parallel. If \( \vec{x} \) and \( u \) are not collinear, \( \vec{x} \) cannot be written as a scalar multiple of \( u \).

Two vectors \( \vec{x} \) and \( u \) are collinear if and only if it is possible to find a non-zero scalar \( a \) such that \( \vec{x} = au \).

EXAMPLE 1

If possible, express \( \vec{x} \) as a scalar multiple of \( u \).

\[
a. \quad \vec{x} = 4\hat{i} - 8\hat{j} \\
\quad \vec{u} = 6\hat{i} - 12\hat{j}
\]

\[
b. \quad \vec{x} = (10, -8, 3) \\
\quad \vec{u} = (5, -4, 6)
\]

Solution

\( \text{a. Let } a \text{ be a scalar.} \)
\[
\text{If } \quad \vec{x} = au \\
\text{Then } \quad (4\hat{i} - 8\hat{j}) = a(6\hat{i} - 12\hat{j}) \\
\quad 4\hat{i} - 8\hat{j} = 6a\hat{i} - 12a\hat{j}
\]
If the vector on the left is equal to the vector on the right, then the coefficients of \( \hat{i} \) and \( \hat{j} \) on the left must be the same as the corresponding coefficients on the right.

Then \( 4 = 6a \) and \( -8 = -12a \)

\[
\begin{align*}
a &= \frac{2}{3} \\
a &= \frac{2}{3}
\end{align*}
\]

The value for \( a \) in the two cases is consistent.

Thus \( \overrightarrow{x} = \frac{2}{3}\overrightarrow{u} \) and \( \overrightarrow{x} \) and \( \overrightarrow{u} \) are collinear.

b. Proceed as in a.

If \( \overrightarrow{x} = a\overrightarrow{u} \),

then \( (10, -8, 3) = a(5, -4, 6) \)

\( (10, -8, 3) = (5a, -4a, 6a) \).

Equating corresponding components,

\[
\begin{align*}
10 &= 5a \\
-8 &= -4a \\
3 &= 6a
\end{align*}
\]

\[
\begin{align*}
a &= 2 \\
a &= 2 \\
a &= \frac{1}{2}
\end{align*}
\]

Since the scalar \( a \) must be the same for all components of the vectors, there is no scalar multiple of \( \overrightarrow{u} \) which equals \( \overrightarrow{x} \). These two vectors are non-collinear.

Consider now a more complicated situation.

Under what circumstances is it possible to express a given vector \( \overrightarrow{x} \) in terms of two vectors \( \overrightarrow{u} \) and \( \overrightarrow{v} \)? The answer is that you can write \( \overrightarrow{x} = a\overrightarrow{u} + b\overrightarrow{v} \) and determine numerical values for the scalars \( a \) and \( b \) only when the three vectors \( \overrightarrow{x}, \overrightarrow{u}, \) and \( \overrightarrow{v} \) are coplanar. The equation \( \overrightarrow{x} = a\overrightarrow{u} + b\overrightarrow{v} \) implies that the three vectors \( \overrightarrow{x}, a\overrightarrow{u}, \) and \( b\overrightarrow{v} \) form a triangle.

If \( \overrightarrow{x} \) does not lie in the plane of \( \overrightarrow{u} \) and \( \overrightarrow{v} \), this triangle cannot exist.

Three vectors \( \overrightarrow{x}, \overrightarrow{u}, \) and \( \overrightarrow{v} \) are coplanar if and only if it is possible to find non-zero scalars \( a \) and \( b \) such that \( \overrightarrow{x} = a\overrightarrow{u} + b\overrightarrow{v} \).

**Example 2**

Three vectors \( \overrightarrow{u}, \overrightarrow{v}, \) and \( \overrightarrow{x} \), have magnitudes \( |\overrightarrow{u}| = 10, |\overrightarrow{v}| = 15, \) and \( |\overrightarrow{x}| = 24 \).

If \( \overrightarrow{x} \) lies between \( \overrightarrow{u} \) and \( \overrightarrow{v} \) in the same plane, making an angle of 20° with \( \overrightarrow{u} \) and 30° with \( \overrightarrow{v} \), express \( \overrightarrow{x} \) as a linear combination of \( \overrightarrow{u} \) and \( \overrightarrow{v} \).

**Solution**

You are to find scalar multiples of \( \overrightarrow{u} \) and \( \overrightarrow{v} \), the sum of which equals \( \overrightarrow{x} \). Let \( \overrightarrow{x} = a\overrightarrow{u} + b\overrightarrow{v} \), where \( a \) and \( b \) are coefficients to be determined.
Make a parallelogram by drawing lines from the tip of \( \vec{x} \) in the vector diagram, parallel to \( \vec{u} \) and \( \vec{v} \). The sides of this parallelogram are the required vectors \( a\vec{u} \) and \( b\vec{v} \), which add to \( \vec{x} \). In this particular case, \( a\vec{u} \) is longer than \( \vec{u} \), so you should expect \( a \) to be greater than one. On the other hand, \( b\vec{v} \) is shorter than \( \vec{v} \), so \( b \) will be less than one.

Now draw the triangle for the addition of the vectors.

From \( \frac{a |\vec{u}|}{\sin 30^\circ} = \frac{|\vec{x}|}{\sin 130^\circ} \)

\[ a = \frac{24 \sin 30^\circ}{10 \sin 130^\circ} \]

\[ a \approx 1.566 \]

and from \( \frac{b |\vec{v}|}{\sin 20^\circ} = \frac{|\vec{x}|}{\sin 130^\circ} \)

\[ b = \frac{24 \sin 20^\circ}{15 \sin 130^\circ} \]

\[ b \approx 0.714 \]

Therefore, \( \vec{x} = 1.566\vec{u} + 0.714\vec{v} \), and \( \vec{x} \) is expressed as a linear combination of \( \vec{u} \) and \( \vec{v} \). Note that \( a \) is greater than one and \( b \) is less than one, as expected.

**EXAMPLE 3**

Determine whether or not the three vectors \( \vec{u} = (3, -1, 4) \), \( \vec{v} = (6, -4, -8) \), and \( \vec{x} = (7, -3, 4) \) are coplanar. If they are, express \( \vec{x} \) as a linear combination of \( \vec{u} \) and \( \vec{v} \).

**Solution**

We could calculate the triple scalar product. If it is zero, the vectors are coplanar, but we would be left with the problem of expressing \( \vec{x} \) as a linear combination of \( \vec{u} \) and \( \vec{v} \). Instead, we proceed as follows.

Write \( \vec{x} = a\vec{u} + b\vec{v} \). If we can find values of \( a \) and \( b \), the vectors are coplanar. If we cannot find values for \( a \) and \( b \), the vectors are not coplanar. We have

\[(7, -3, 4) = a(3, -1, 4) + b(6, -4, -8)\]

\[= (3a + 6b, -a - 4b, 4a - 8b)\]

Equating components, we get the system of equations

\[3a + 6b = 7\]
\[-a - 4b = -3\]
\[4a - 8b = 4\]
We solve any two of these equations, say the first and second.

\[
\begin{align*}
3a + 6b &= 7 \\
-3a - 12b &= -9 \\
-6b &= -2
\end{align*}
\]

so \( b = \frac{1}{3} \)

Substituting into either of the equations, we find \( a = \frac{5}{3} \).

We now see if these values satisfy the third equation

\[
4\left(\frac{5}{3}\right) - 8\left(\frac{1}{3}\right) = \frac{20}{3} - \frac{8}{3} = 4, \text{ as required}
\]

Therefore the vectors are coplanar, and \( \vec{x} = \frac{5}{3} \vec{u} + \frac{1}{3} \vec{v} \).

**Exercise 6.1**

**Part A**

**Communication**

1. Explain why it is impossible to express \( \vec{x} \) as a linear combination of \( \vec{u} \) and \( \vec{v} \), when \( \vec{x} \) does not lie in the plane of \( \vec{u} \) and \( \vec{v} \).

2. The vectors \( \vec{u}, \vec{v}, \) and \( \vec{x} \) are any non-zero vectors in the xy-plane. Is it always possible to express \( \vec{x} \) as a linear combination of \( \vec{u} \) and \( \vec{v} \)? Explain.

3. What information does the cross product of \( \vec{u} \) and \( \vec{x} \) give about the collinearity of \( \vec{u} \) and \( \vec{x} \)?

**Knowledge/Understanding**

4. Write each of the following vectors as a linear combination of \( \hat{i} \) and \( \hat{j} \).
   a. the vector \( \vec{p} = (-4, 5) \)
   b. the position vector of the point \( A(8, -3) \)
   c. a vector directed at an angle of 45º with a magnitude of \( \sqrt{2} \)
   d. a vector directed at an angle of 150º with a magnitude of 6

5. a. Can every vector in the xy-plane be written as a linear combination of \( \vec{u} = (1, 4) \) and \( \vec{v} = (-2, 5) \)? Justify your answer.
   b. Write the vector \( (-567, -669) \) in terms of \( \vec{u} \) and \( \vec{v} \).
6. Can every vector in the \(xy\)-plane be written as a linear combination of \( \mathbf{u} = (-4, -6) \) and \( \mathbf{v} = (10, 15) \)? Justify your answer.

**Part B**

### Knowledge/Understanding

7. Are the following sets of vectors coplanar?

    a. \((1, -1, 1), (0, 1, 1), (1, 0, 2)\)
    b. \((1, 0, 1), (1, 1, 1), (1, 0, -1)\)

### Application

8. If \( \mathbf{u} = (2, 1, 1) \) and \( \mathbf{v} = (-1, 1, 3) \)

    a. which of the following vectors can be written in the form \( s\mathbf{u} + t\mathbf{v} \)
        
        (i) \((4, 2, 2)\)
        (ii) \((1, 2, 4)\)
        (iii) \((1, 5, 11)\)
        (iv) \((4, 5, 8)\)

    b. Find another vector that can be expressed in the form \( s\mathbf{u} + t\mathbf{v} \).

    c. Find another vector which *cannot* be expressed in the form \( s\mathbf{u} + t\mathbf{v} \), and explain why it cannot.

### Application

9. Given that

   \[
   \begin{align*}
   \mathbf{u} &= x\mathbf{a} + 2y\mathbf{b} \\
   \mathbf{v} &= -2y\mathbf{a} + 3y\mathbf{b} \\
   \mathbf{w} &= 4\mathbf{a} - 2\mathbf{b}
   \end{align*}
   \]

   where \( \mathbf{a} \) and \( \mathbf{b} \) are not collinear, find the values of \( x \) and \( y \) for which \( 2\mathbf{u} - \mathbf{v} = \mathbf{w} \).

**Part C**

10. Find values of \( a, b, \) and \( c \) which satisfy each of the following equations.

    a. \( a(2, 1, 0) + b(-3, 4, 5) + c(2, 0, 3) = (-4, 10, 7) \)
    b. \( a(3, -1, 2) + b(-1, 1, 3) + c(2, 1, 5) = (2, 5, 16) \)

### Application

11. a. Demonstrate that the three vectors \( \mathbf{u} = (1, 3, 2), \mathbf{v} = (1, -1, 1), \) and \( \mathbf{w} = (5, 1, -4) \) are mutually perpendicular.

    b. Express each of the vectors \( \hat{i}, \hat{j}, \) and \( \hat{k} \) as a linear combination of the vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{w} \).

12. If \( \mathbf{u} = (5, -5, 2), \mathbf{v} = (1, 8, -4), \) and \( \mathbf{w} = (-2, -1, 2) \), express \( \mathbf{x} = (-3, 6, 8) \)

    a. in terms of \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \)
    b. in terms of the unit vectors \( \hat{u}, \hat{v}, \) and \( \hat{w} \)

### Thinking/Inquiry/Problem Solving

13. Vectors \( \mathbf{u}, \mathbf{v}, \) and \( \mathbf{x} \) in the \(xy\)-plane make angles of 20\(^\circ\), 50\(^\circ\), and 130\(^\circ\), respectively, with the \(x\)-axis. If \( |\mathbf{u}| = 2, |\mathbf{v}| = 10, \) and \( |\mathbf{x}| = 4 \)

    a. express \( \mathbf{x} \) as a linear combination of \( \mathbf{u} \) and \( \mathbf{v} \)
    b. express \( \mathbf{x} \) as a linear combination of the unit vectors \( \hat{u} \) and \( \hat{v} \)
Section 6.2 — Linear Dependence and Independence

The concepts of linear dependence and independence are fundamental in vector algebra. Their importance is both theoretical and practical. Two vectors are linearly dependent if they are collinear or parallel. Three vectors are linearly dependent if they are coplanar.

Let us first consider just two vectors. For example, suppose \( \vec{u} = 2\vec{v} \). This statement can be rewritten as \( \vec{u} - 2\vec{v} = \vec{0} \).

Geometrically, this means that multiplying \( \vec{v} \) by the scalar \(-2\) and adding it to \( \vec{u} \) brings us back to the zero vector \( \vec{0} \).

Parallel vectors

Non-parallel vectors

We can see that whenever two vectors \( \vec{u} \) and \( \vec{v} \) are parallel, there is a relationship \( au + bv = \vec{0} \), where \( a \) and \( b \) are not both zero. We then say that \( \vec{u} \) and \( \vec{v} \) are linearly dependent vectors.

On the other hand, consider two vectors \( \vec{m} \) and \( \vec{n} \) that are not parallel. There is no possible way to combine multiples of these vectors so that \( am + bn = \vec{0} \), unless \( a = b = 0 \). These vectors are called linearly independent vectors.

### Linear Dependence of Two Vectors

Two vectors \( \vec{u} \) and \( \vec{v} \) are called **linearly dependent** if and only if there are scalars \( a \) and \( b \), not both zero, such that

\[
au + bv = \vec{0}.
\]

Two vectors are **linearly independent** if they are not linearly dependent.

### Linear Dependence of Three Vectors

Three vectors \( \vec{u} \), \( \vec{v} \), and \( \vec{w} \) are **linearly dependent** if and only if there are scalars \( a \), \( b \), and \( c \), not all zero, such that

\[
au + bv + cw = \vec{0}.
\]

Three vectors are **linearly independent** if they are not linearly dependent.
EXAMPLE 1

a. Demonstrate that the three vectors \( \vec{u} = (2, 1), \vec{v} = (-1, 7), \) and \( \vec{w} = (-4, 3) \) are linearly dependent by showing that it is possible to find coefficients \( a, b, \) and \( c, \) not all zero, such that the linear combination \( a\vec{u} + b\vec{v} + c\vec{w} \) is equal to the zero vector.

b. Express each vector as a linear combination of the other two.

Solution

a. If the vectors are linearly dependent, then there are values \( a, b, c \) such that

\[
\begin{align*}
\vec{u} & = (2, 1), \\
\vec{v} & = (-1, 7), \\
\vec{w} & = (-4, 3),
\end{align*}
\]

\( a\vec{u} + b\vec{v} + c\vec{w} = \vec{0} \)

\[
\begin{align*}
2a - b - 4c & = 0 \\
7a + 7b + 3c & = 0
\end{align*}
\]

The components of the vector on the left must be zero.

\[
\begin{align*}
2a - b - 4c & = 0 \\
a + 7b + 3c & = 0
\end{align*}
\]

Since there are three variables but only two equations, it is not possible to find unique values for \( a, b, \) and \( c, \)

eliminating \( a \)

\[
\begin{align*}
2a - b - 4c & = 0 \\
-2a - 14b - 6c & = 0 \\
-15b - 10c & = 0
\end{align*}
\]

\[
\therefore b = -\frac{2}{3}c
\]

eliminating \( b \)

\[
\begin{align*}
2a - b - 4c & = 0 \\
a + 7b + 3c & = 0 \\
-15a - 25c & = 0
\end{align*}
\]

\[
\therefore a = \frac{5}{3}c
\]

To avoid fractions, we choose \( c = 3, \) whereupon \( b = -2 \) and \( a = 5. \)

Consequently,

\[
5(2, 1) - 2(-1, 7) + 3(-4, 3) = \vec{0}
\]

or

\[
5\vec{u} - 2\vec{v} + 3\vec{w} = \vec{0}
\]

Thus there are values of \( a, b, \) and \( c, \) not all zero, which make the linear combination \( a\vec{u} + b\vec{v} + c\vec{w} \) equal to the zero vector. Then the three vectors \( \vec{u}, \vec{v}, \) and \( \vec{w} \) are linearly dependent.

b. If \( 5\vec{u} - 2\vec{v} + 3\vec{w} = \vec{0}, \) then

\[
\begin{align*}
\vec{u} & = \frac{2}{5}\vec{v} - \frac{3}{5}\vec{w}, \\
\vec{v} & = \frac{5}{2}\vec{u} + \frac{3}{2}\vec{w},
\end{align*}
\]

\( \text{and} \quad \vec{w} = -\frac{5}{3}\vec{u} + \frac{2}{3}\vec{v} \)

EXAMPLE 2

Prove that three non-collinear vectors \( \vec{u}, \vec{v}, \) and \( \vec{w} \) are linearly dependent if and only if they are coplanar.
Solution

First, suppose that \( \vec{u}, \vec{v}, \text{and} \vec{w} \) are linearly dependent, so that \( a\vec{u} + b\vec{v} + c\vec{w} = \vec{0} \) for some scalars \( a, b, c, \) not all zero. Assume that \( c \) is one of the non-zero scalars. It is then possible to divide by \( c \) and solve for \( \vec{w} \).

\[
\vec{w} = -\frac{a}{c} \vec{u} - \frac{b}{c} \vec{v}
\]

Thus, \( \vec{w} \) is expressed as a linear combination of \( \vec{u} \) and \( \vec{v} \), so \( \vec{w} \) must lie in the plane of \( \vec{u} \) and \( \vec{v} \), making the three vectors coplanar.

Conversely, suppose that \( \vec{u}, \vec{v}, \text{and} \vec{w} \) are coplanar. Then \( \vec{w} \), for example, can be written as a linear combination of \( \vec{u} \), and \( \vec{v} \) as \( \vec{w} = a\vec{u} + b\vec{v} \). Hence, \( a\vec{u} + b\vec{v} - \vec{w} = \vec{0} \). Because the coefficients \( a, b, \text{and} -1 \) are not all zero, the three vectors must be linearly dependent.

We can always form a linear combination of vectors. For instance, we can always write an equation of the form \( a\vec{u} + b\vec{v} = \vec{0} \) for any pair of vectors \( \vec{u} \) and \( \vec{v} \). If the two vectors are non-collinear, however, the only values of \( a \) and \( b \) that can make the equation true are \( a = 0 \) and \( b = 0 \), for then, \( 0\vec{u} + 0\vec{v} = \vec{0} \).

We can make a similar observation about a linear combination of three vectors. We can always write an equation of the form \( a\vec{x} + b\vec{u} + c\vec{v} = \vec{0} \). But if the three vectors are not coplanar, and no two are collinear, then it is impossible to express one of the vectors in terms of the other two. Under these circumstances, the only values of \( a, b, \text{and} c \) that can make the equation true are \( a = 0, b = 0, \text{and} c = 0 \).

A linear combination of vectors will, of course, equal the zero vector if we set all the coefficients \( a, b, c, \ldots \) equal to zero. However, if making coefficients equal zero is the only way the linear combination can equal the zero vector, then the vectors cannot be dependent. Such vectors are said to be linearly independent.

A set of vectors \( \vec{u}, \vec{v}, \vec{w}, \vec{x}, \ldots \) is linearly independent if the only linear combination \( a\vec{u} + b\vec{v} + c\vec{w} + d\vec{x} + \ldots \) that produces the zero vector is the one in which the coefficients \( a, b, c, d, \ldots \) are all zero.
EXAMPLE 3

Prove that \( \hat{i} \) and \( \hat{j} \), the basis vectors for a plane, are linearly independent.

**Solution**

Start by forming a linear combination of \( \hat{i} \) and \( \hat{j} \) and setting it equal to zero.

\[
a\hat{i} + b\hat{j} = \vec{0}
\]

Either \( a = 0 \) or \( a \neq 0 \). Suppose that \( a \neq 0 \). Then \( \hat{i} = -\frac{b}{a}\hat{j} \).

If \( b \neq 0 \), \( \hat{i} \) must be a scalar multiple of \( \hat{j} \).
If \( b = 0 \), \( \hat{i} \) must equal the zero vector.

Both of these statements are false.

Consequently, the assumption \( a \neq 0 \) is impossible, so \( a \) must be zero. It follows that \( 0\hat{i} + b\hat{j} = \vec{0} \), which means that \( b = 0 \), since \( \hat{j} \neq \vec{0} \). Thus, the values \( a = 0 \) and \( b = 0 \) are the only values of \( a \) and \( b \) for which the linear combination \( a\hat{i} + b\hat{j} = \vec{0} \). Therefore, the vectors \( \hat{i} \) and \( \hat{j} \) must be linearly independent.

EXAMPLE 4

Show that \( \vec{u} = 4\hat{i} + 8\hat{j} \) and \( \vec{v} = 6\hat{i} - 3\hat{j} \) are linearly independent.

**Solution**

For scalars \( a \) and \( b \), let

\[
a\vec{u} + b\vec{v} = \vec{0}
\]

\[
a(4\hat{i} + 8\hat{j}) + b(6\hat{i} - 3\hat{j}) = \vec{0}
\]

\[
4a\hat{i} + 8a\hat{j} + 6b\hat{i} - 3b\hat{j} = \vec{0}
\]

\[
(4a + 6b)\hat{i} + (8a - 3b)\hat{j} = \vec{0}
\]

Since \( \hat{i} \) and \( \hat{j} \) are linearly independent vectors, both coefficients are equal to zero.

\[
4a + 6b = 0
\]

\[
8a - 3b = 0
\]

Thus,

\[
a = b = 0
\]

Then \( \vec{u} \) and \( \vec{v} \) are linearly independent.

EXAMPLE 5

Given that the vectors \( \vec{u} \) and \( \vec{v} \) are linearly independent and \( (3 - s)\vec{u} + t\vec{v} = 5\vec{u} - 4s\vec{v} \), determine the values of \( s \) and \( t \).

**Solution**

\[
(3 - s)\vec{u} + t\vec{v} = 5\vec{u} - 4s\vec{v}
\]

\[
(3 - s)\vec{u} - 5\vec{u} + t\vec{v} + 4s\vec{v} = \vec{0}
\]

\[
(-2 - s)\vec{u} + (t + 4s)\vec{v} = \vec{0}
\]
Since \( \vec{u} \) and \( \vec{v} \) are given to be linearly independent, the coefficients of \( \vec{u} \) and \( \vec{v} \) must be zero. Therefore,

\[
-2 - s = 0 \quad \text{and} \quad t + 4s = 0
\]

\[
s = -2 \quad \text{and} \quad t = 8
\]

Two vectors are linearly dependent if they are collinear. Any two non-collinear vectors define a plane. Any pair of linearly independent vectors can be designated as basis vectors for a plane. Every vector in the plane can be expressed as a linear combination of these basis vectors. Thus, the vectors \( \hat{i} \) and \( \hat{j} \) are not the only vectors that could be used as basis vectors for a plane. They are just a particularly simple and convenient choice.

Three vectors are linearly dependent if they are coplanar. Three vectors are coplanar if \( (\vec{u} \times \vec{v}) \cdot \vec{w} = 0 \), since \( (\vec{u} \times \vec{v}) \) is perpendicular to the plane of \( \vec{u} \) and \( \vec{v} \), and if \( \vec{w} \) is perpendicular to this cross product, it must lie in the plane of \( \vec{u} \) and \( \vec{v} \). On the other hand, if the triple scalar product of three vectors is not zero, then the three vectors are non-coplanar. Therefore, they are linearly independent. As a consequence, they can be designated as basis vectors for space, in terms of which every three-dimensional vector can be expressed.

### Exercise 6.2

**Part A**

1. Given that \( \vec{w} = a\vec{u} + b\vec{v} \), what can be said about \( \vec{w} \)
   a. if \( \vec{u} \) and \( \vec{v} \) are linearly independent?
   b. if \( \vec{u} \) and \( \vec{v} \) are linearly dependent?

2. a. Are three vectors lying in a plane always linearly dependent? Explain.
   b. Given three vectors in a plane, under what circumstances is it impossible to express one of them as a linear combination of the other two?
   c. Can any set of three non-collinear vectors be used as a basis for space? Explain.

3. If \( \vec{u} \) and \( \vec{v} \) are linearly independent vectors, find the values of \( s \) and \( t \) for each of the following equations.
   a. \( s\vec{u} + 2\vec{v} = \vec{0} \)
   b. \( (s + 5)\vec{u} + (t - 3)\vec{v} = \vec{0} \)
   c. \( (s - 2)\vec{u} = (s - t - 3)\vec{v} \)
   d. \( s\vec{u} + 7\vec{v} = 5\vec{u} - t\vec{v} \)
4. If $\vec{u}$, $\vec{v}$, and $\vec{w}$ are linearly independent vectors, find the values of $r$, $s$, and $t$ for each of the following equations.
   a. $ru + (2s - 1)v + (r + s + t)w = \vec{0}$
   b. $(r - s - 5)u + (r + s + 1)v + (r + st)w = \vec{0}$

5. Given that the vectors $\vec{u}$ and $\vec{v}$ are linearly independent, determine the value of $k$, if possible, for each of the following equations.
   a. $(k + 2)u + (k - 2)v = \vec{0}$
   b. $(6k - 4)u + (8 - 12k)v = \vec{0}$
   c. $(k^2 - 4)u + (k + 2)v = \vec{0}$
   d. $ku + 3v = \vec{0}$

Part B

6. Given that $\vec{u}$ and $\vec{v}$ are linearly independent vectors and $a$ and $b$ are non-zero scalars, prove that $au$ and $bv$ are linearly independent vectors.

7. Show that the representation of a three-dimensional vector in terms of $\hat{i}$, $\hat{j}$, and $\hat{k}$ is unique.

8. Show that the vectors $(-1, 1, 1)$, $(1, -1, 1)$ and $(1, 1, -1)$ can be used as a basis for vectors in space.

9. Show that the vectors $\vec{a}$, $\vec{a} \times \vec{b}$, and $\vec{a} \times (\vec{a} \times \vec{b})$ can be used as a basis for vectors in space, where $\vec{a}$ and $\vec{b}$ are any non-collinear vectors.

10. Determine whether the following sets of vectors form bases for two-dimensional space. If a set forms a basis, determine the coordinates of $\vec{v} = (8, 7)$ relative to this base.
    a. $\vec{v}_1 = (1, 2)$, $\vec{v}_2 = (3, 5)$
    b. $\vec{v}_1 = (3, 5)$, $\vec{v}_2 = (6, 10)$

11. Determine whether the following sets of vectors form bases for three-dimensional space. If a set forms a basis, determine the coordinates of $\vec{v} = (1, 2, 3)$ relative to the basis.
    a. $\vec{v}_1 = (-1, 0, 1)$, $\vec{v}_2 = (2, 1, 1)$, $\vec{v}_3 = (3, 1, 1)$
    b. $\vec{v}_1 = (1, 3, -1)$, $\vec{v}_2 = (2, 1, 1)$, $\vec{v}_3 = (-4, 3, -5)$
    c. $\vec{v}_1 = (1, 0, 0)$, $\vec{v}_2 = (1, 1, 0)$, $\vec{v}_3 = (1, 1, 1)$
Part C

12. If \((3\vec{u} + 4\vec{v})\) and \((6\vec{u} - 2\vec{v})\) are linearly independent vectors, show that \(\vec{u}\) and \(\vec{v}\) must be linearly independent also.

13. The vectors \(\vec{a}\), \(\vec{b}\), and \(\vec{c}\) are basis vectors for space. \(\vec{d}\) is any three-dimensional vector, and \(\vec{d} = k\vec{a} + l\vec{b} + m\vec{c}\). Show that this representation of \(\vec{d}\) in terms of \(\vec{a}\), \(\vec{b}\), and \(\vec{c}\) is unique.

14. If \(\vec{u}\), \(\vec{v}\), and \(\vec{w}\) are mutually perpendicular, linearly independent vectors, are the vectors \(\vec{u} + \vec{v}\), \(\vec{v} + \vec{w}\), and \(\vec{w} + \vec{u}\) linearly dependent or linearly independent?

15. The vectors \(\vec{u}\) and \(\vec{v}\) are linearly independent. Find \(s\), if the vectors \((1 - s)\vec{u} - \frac{2}{3}\vec{v}\) and \(3\vec{u} + s\vec{v}\) are parallel.
Section 6.3 — Division of a Line Segment

Two points determine a straight line. If a third point lies on this line, the three points are said to be collinear. When three points, \( A, B, \) and \( P \) are collinear, two of the points, say \( A \) and \( B \), are taken as the end points of a line segment. The third point \( P \) is called a division point of the segment.

When \( P \) lies between \( A \) and \( B \) on the line segment, then \( P \) is said to divide the segment \textit{internally}. When points, such as \( Q \) or \( R \) in the given diagram, lie on an extension of the segment outside the interval \( AB \), then they are said to divide the segment \textit{externally}.

The midpoint of a line segment is an example of an internal division point. The midpoint \( M \) of a segment \( AB \) lies between \( A \) and \( B \) and divides the segment exactly in half. This means that the vectors \( \overrightarrow{AM} \) and \( \overrightarrow{MB} \) are equal. From this equality, a formula for the position vector of the midpoint can be found.

The position vectors of the points \( A, B, \) and \( M \) relative to some origin \( O \) are, respectively, \( \overrightarrow{OA}, \overrightarrow{OB}, \) and \( \overrightarrow{OM} \).

Since \( \overrightarrow{AM} = \overrightarrow{MB} \)

Then \( \overrightarrow{OM} - \overrightarrow{OA} = \overrightarrow{OB} - \overrightarrow{OM} \)

\[ 2\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{OB} \]

Therefore, \( \overrightarrow{OM} = \frac{1}{2} \overrightarrow{OA} + \frac{1}{2} \overrightarrow{OB} \)

The position vector of the midpoint is a linear combination of the position vectors of the endpoints of the line segment. The derivation of this midpoint formula is valid for vectors in both two and three dimensions.

\textbf{EXAMPLE 1}

Find the midpoint of the line segment from \( A(10, -7, -4) \) to \( B(8, 1, -6) \).

\textbf{Solution}

The position vectors of points \( A \) and \( B \) are \( \overrightarrow{OA} = (10, -7, -4) \) and \( \overrightarrow{OB} = (8, 1, -6) \). Therefore,

\[ \overrightarrow{OM} = \frac{1}{2}[(10, -7, -4) + (8, 1, -6)] \]

\[ = (9, -3, -5) \]

The midpoint \( M \) is \((9, -3, -5)\).
There are several equivalent ways to describe the division point of a line segment. The midpoint, for instance, can be spoken of as a ratio or written as a fraction.

\[ \frac{AM}{MB} = \frac{1}{1} \]

In a similar manner,

if \( P \) divides \( AB \) in the ratio 1:3 or \[ \frac{AP}{PB} = \frac{1}{3} \]

then the points must be arranged on the line as in this division-point diagram:

In reading or making a division-point diagram, treat segments going in the same direction as the given segment as positive, and in the opposite direction as negative. Here, the segments \( AP \) and \( PB \) are both positive.

In the case of an external division of a line segment, such as that shown in this division-point diagram,

\[ \frac{AP}{PB} = \frac{3}{-1} \]

You can distinguish an external from an internal division by a negative sign in the ratio or fraction. The negative sign is more conveniently placed on the smaller term of the ratio. In the diagram above, \( P \) divides \( BA \) in the ratio \(-1:3\) since \( BP \) has direction opposite to \( BA \).

**EXAMPLE 2**

Points \( P \) and \( Q \) lie on line segment \( AB \), such that \( AP = 6 \) units, \( PQ = 5 \) units, and \( QB = 3 \) units. In what ratio does

a. \( P \) divide \( AB \)?  

b. \( A \) divide \( BP \)?  

c. \( B \) divide \( PQ \)?  

d. \( A \) divide \( QB \)?

**Solution**

Draw a division-point diagram containing the given information:

a. \( \frac{AP}{PB} = \frac{6}{8} \), so \( P \) divides \( AB \) in the ratio 6:8 or 3:4  

b. \( \frac{BA}{AP} = \frac{14}{-6} \), so \( A \) divides \( BP \) in the ratio 7:−3  

c. \( \frac{PB}{BQ} = \frac{8}{-3} \), so \( B \) divides \( PQ \) in the ratio 8:−3  

d. \( \frac{QA}{AB} = \frac{-11}{14} \), so \( A \) divides \( QB \) in the ratio −11:14
EXAMPLE 3
The points $P$, $Q$, $R$, and $S$ are collinear. $S$ divides $QR$ in the ratio $-3:4$. $R$ divides $SP$ in the ratio $5:1$. In what ratio does $Q$ divide $SP$?

Solution
Write the fractions and draw the division-point diagrams for the two given ratios.

$$\frac{QS}{SR} = \frac{-3}{4}$$

$$\frac{SR}{RP} = \frac{5}{1}$$

The numbers in the ratios are not the actual lengths of the segments. Think of them as the number of equal parts into which the segment has been divided. The points $S$ and $R$ are common to both segments. To compare the ratios, $SR$ must be divided into the same number of parts. Therefore, multiply the first ratio by 5 and the second by 4, and then arrange all the points on a single line.

This makes it clear that $QR$ must contain 5 parts, and therefore that

$$\frac{SQ}{QP} = \frac{15}{9}$$

so $Q$ divides $SP$ in the ratio $15:9$ or $5:3$.

EXAMPLE 4
The point $P$ divides the line segment $AB$ in the ratio $-2:5$. Express $OP$ as a linear combination of $OA$ and $OB$.

Solution
$P$ divides $AB$ in the ratio $-2:5$, so $\frac{AP}{PB} = \frac{-2}{5}$.

Therefore,

$$\overrightarrow{AP} = -\frac{2}{5}\overrightarrow{PB}$$

$$5(\overrightarrow{OP} - \overrightarrow{OA}) = -2(\overrightarrow{OB} - \overrightarrow{OP})$$

$$5\overrightarrow{OP} - 5\overrightarrow{OA} = -2\overrightarrow{OB} + 2\overrightarrow{OP}$$

$$3\overrightarrow{OP} = 5\overrightarrow{OA} - 2\overrightarrow{OB}$$

$$\overrightarrow{OP} = \frac{5}{3}\overrightarrow{OA} - \frac{2}{3}\overrightarrow{OB}$$
The formula for the position vector of the division point found in Example 4 can be generalized. If $P$ divides $AB$ in some ratio $a:b$, then $\frac{AP}{PB} = \frac{a}{b}$.

Therefore,

$$\overrightarrow{AP} = \frac{a}{b}\overrightarrow{PB}$$

$$b(\overrightarrow{OP} - \overrightarrow{OA}) = a(\overrightarrow{OB} - \overrightarrow{OP})$$

$$b\overrightarrow{OP} - b\overrightarrow{OA} = a\overrightarrow{OB} - a\overrightarrow{OP}$$

$$(a + b)\overrightarrow{OP} = b\overrightarrow{OA} + a\overrightarrow{OB}$$

$$\overrightarrow{OP} = \frac{b}{a + b}\overrightarrow{OA} + \frac{a}{a + b}\overrightarrow{OB}$$

**Division-Point Theorem**

Points $A$, $B$, and $P$ are collinear if and only if

$$\overrightarrow{OP} = \frac{b}{a + b}\overrightarrow{OA} + \frac{a}{a + b}\overrightarrow{OB}.$$ 

This theorem shows that when three points are collinear, their position vectors are linearly dependent and, hence, coplanar.

**EXAMPLE 5**

Prove that if $\overrightarrow{OP} = k\overrightarrow{OA} + l\overrightarrow{OB}$, and $k + l = 1$, this is sufficient to guarantee that the points $A$, $B$, and $P$ are collinear.

**Solution**

Since $k + l = 1$, $k = 1 - l$

Then

$$\overrightarrow{OP} = (1 - l)\overrightarrow{OA} + l\overrightarrow{OB}$$

$$= \overrightarrow{OA} - l\overrightarrow{OA} + l\overrightarrow{OB}$$

So

$$\overrightarrow{OP} - \overrightarrow{OA} = l(\overrightarrow{OB} - \overrightarrow{OA})$$

$$\overrightarrow{AP} = l\overrightarrow{AB}$$

Therefore, the vectors $\overrightarrow{AP}$ and $\overrightarrow{AB}$ are parallel, and so $A$, $B$, and $P$ are collinear. This proves the *if* part of the division-point theorem. The derivation of the division-point formula shows it is necessary for collinearity, and constitutes a proof of the *only if* part of the theorem.
Part A

1. Can a line segment be divided in the ratio 1:1? Explain.

2. Points \( A, B, C, \) and \( D \) are located on a line as shown in the given diagram. Determine
   a. the ratio in which \( C \) divides \( AD \)
   b. the ratio in which \( B \) divides \( AD \)
   c. the ratio in which \( A \) divides \( BD \)
   d. the ratio in which \( D \) divides \( AB \)
   e. the ratio in which \( B \) divides \( CD \)

3. Draw a division-point diagram for each of the following statements.
   a. point \( A \) divides \( BC \) in the ratio 2:1
   b. point \( U \) divides \( ST \) in the ratio 3:1
   c. point \( Q \) divides \( PR \) in the ratio 1:2
   d. point \( K \) divides \( MN \) in the ratio 5:8
   e. point \( D \) divides \( EF \) in the ratio 2:3

Part B

4. If the point \( P \) divides \( AB \) in the ratio 1:2 and the point \( Q \) divides \( AB \) in the ratio 1:2,
   a. in what ratio does \( A \) divide \( QB? \)
   b. in what ratio does \( B \) divide \( QP? \)
   c. in what ratio does \( Q \) divide \( AP? \)
   d. in what ratio does \( P \) divide \( QA? \)
   e. in what ratio does \( B \) divide \( PA? \)

5. If \( T \) divides \( AB \) in the ratio 2:1, prove from first principles that \( \frac{OT}{OT} = 2OB - OA \).
6. If \( \overrightarrow{OB} = \frac{2}{3} \overrightarrow{OC} + \frac{1}{3} \overrightarrow{OD} \), prove from first principles that \( B, C, \) and \( D \) are collinear points.

7. Which statements indicate that \( A, B, \) and \( C \) are collinear points?
   a. \( \overrightarrow{OA} = \frac{3}{4} \overrightarrow{OB} + \frac{1}{4} \overrightarrow{OC} \)
   b. \( \overrightarrow{OC} = \frac{3}{5} \overrightarrow{OA} + \frac{3}{5} \overrightarrow{OB} \)
   c. \( \overrightarrow{OA} = 5 \overrightarrow{OB} - 4 \overrightarrow{OC} \)
   d. \( \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{0} \)

8. In what ratio does \( P \) divide \( AB \) if
   a. \( \overrightarrow{OP} = \frac{2}{9} \overrightarrow{OA} + \frac{7}{9} \overrightarrow{OB} \)
   b. \( \overrightarrow{OP} = \frac{-4}{9} \overrightarrow{OA} + \frac{13}{9} \overrightarrow{OB} \)
   c. \( \overrightarrow{OP} = 5 \overrightarrow{OA} - 4 \overrightarrow{OB} \)
   d. \( \overrightarrow{OP} = \frac{9}{7} \overrightarrow{OA} - \frac{2}{7} \overrightarrow{OB} \)

9. Express \( \overrightarrow{OA} \) as a linear combination of \( \overrightarrow{OB} \) and \( \overrightarrow{OC} \), when \( A \) divides \( BC \) in the given ratios.
   a. 3:2
   b. \(-2:3\)
   c. 3:2

10. Find the midpoint of the line segment joining \( A(3, 4, 6) \) to \( B(7, 8, -3) \).

11. Find the points that trisect the line segment from \( A(3, 6, 8) \) to \( B(6, 0, -1) \).

12. \( A(2, 10) \) and \( B(1, -5) \) are the endpoints of \( AB \). Find the point that divides \( AB \) in each of the given ratios.
   a. 1:5
   b. \( 2:-1 \)
   c. \(-4:7\)
   d. 3:12

Part C

13. If \( \overrightarrow{OE} = \frac{-2}{3} \overrightarrow{OD} + \frac{7}{3} \overrightarrow{OF} \) and \( \overrightarrow{OG} = \frac{1}{3} \overrightarrow{OD} + \frac{4}{3} \overrightarrow{OF} \),
   a. in what ratio does \( D \) divide \( GE \)?
   b. in what ratio does \( F \) divide \( GE \)?

14. If \( P \) divides \( AB \) in the ratio \( a:b \), find the ratio of the areas of the triangles \( OAP \) and \( OPB \).

15. Prove that if \( \overrightarrow{OD} = r \overrightarrow{OA} + s \overrightarrow{OB} + t \overrightarrow{OC} \) and \( r + s + t = 1 \), the four points \( A, B, C, \) and \( D \) are coplanar.
We need to be able to solve a problem in several different ways. Sometimes one solution may be more direct and easier than another. If we try different methods of solution, we gain insight into the mathematical principles involved and increase our confidence in the results.

Euclidean proofs are the usual way to establish the properties of geometrical figures. The use of vectors is an alternate way to accomplish the same result.

There are two distinct approaches that can be taken when we use vectors to do proofs. One approach is to use point-to-point vectors. The other is to use position vectors. Point-to-point vectors usually lie in the plane of a figure and join one point of the figure to another. *Position vectors*, on the other hand, point from some outside origin, which is not usually part of the figure, to points in the figure.

The two methods are illustrated in Examples 1 and 2 below. Remember that there are several things to do before you can actually start a proof. For instance, the proposition to be proved is usually expressed in words, so your first job is to express what is given and what is to be proved in the form of vector formulas or equations. To do this, you will need a suitably labelled diagram.

**EXAMPLE 1**

Two of the opposite sides of a quadrilateral are parallel and equal in length. Using point-to-point vectors, prove that the other two opposite sides are also parallel and equal in length.

**Solution**

Let $ABCD$ be a quadrilateral in which $AB = CD$ and $AB \parallel CD$. Using vectors, we write $\vec{AB} = \vec{DC}$.

Likewise, what is to be proved can be written $\vec{AD} = \vec{BC}$.

Then

\[
\vec{AD} = \vec{AB} + \vec{BD} = \vec{DC} + \vec{BD} = \vec{BD} + \vec{DC} = \vec{BC}
\]

Therefore, if two of the opposite sides of a quadrilateral are parallel and equal, so are the other two opposite sides.

**EXAMPLE 2**

Two of the opposite sides of a quadrilateral are parallel and equal in length. Using position vectors, prove that the other two opposite sides are also parallel and equal in length.
Solution
Let \(ABCD\) be a quadrilateral having \(AB = CD\) and \(AB \parallel CD\). Let \(O\) be an origin that is not in the plane of the quadrilateral.

As in Example 1, \(\overrightarrow{AB} = \overrightarrow{DC}\) is given, and \(\overrightarrow{AD} = \overrightarrow{BC}\) is to be proved.

Since
\[
\overrightarrow{AB} = \overrightarrow{DC} = \overrightarrow{OD} - \overrightarrow{OA} = \overrightarrow{OC} - \overrightarrow{OD} = \overrightarrow{AD} = \overrightarrow{BC},
\]

The conclusion is the same as that of Example 1.

Sometimes a proof using position vectors requires the use of the division-point formula and the concept of linear independence. An example of that kind of proof is shown next.

**EXAMPLE 3**
Prove that the medians of a triangle intersect at a point that divides each median in the ratio 2:1.

**Solution**
In \(\triangle ABC\), \(D\) and \(E\) are the midpoints of \(BC\) and \(AC\), respectively. If \(O\) is a point not in the plane of the triangle, then
\[
\overrightarrow{OD} = \frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC}, \quad \text{and} \quad \overrightarrow{OE} = \frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OC}.
\]

Let \(\overrightarrow{OG} = k\overrightarrow{OA} + l\overrightarrow{OD}, (k + l = 1)\)

Then
\[
= k\overrightarrow{OA} + l\left(\frac{1}{2}\overrightarrow{OB} + \frac{1}{2}\overrightarrow{OC}\right)
\]
\[
= k\overrightarrow{OA} + \frac{1}{2}l\overrightarrow{OB} + \frac{1}{2}l\overrightarrow{OC}
\]

Similarly \(\overrightarrow{OG} = m\overrightarrow{OB} + n\overrightarrow{OE}, (m + n = 1)\)

\[
= m\overrightarrow{OB} + n\left(\frac{1}{2}\overrightarrow{OA} + \frac{1}{2}\overrightarrow{OC}\right)
\]
\[
= \frac{n}{2}\overrightarrow{OA} + m\overrightarrow{OB} + \frac{n}{2}\overrightarrow{OC}
\]

These two expressions for \(\overrightarrow{OG}\) must be equal. Therefore,

\[
k\overrightarrow{OA} + \frac{l}{2}\overrightarrow{OB} + \frac{l}{2}\overrightarrow{OC} = \frac{n}{2}\overrightarrow{OA} + m\overrightarrow{OB} + \frac{n}{2}\overrightarrow{OC}
\]

or
\[
(k - \frac{n}{2})\overrightarrow{OA} + (\frac{l}{2} - m)\overrightarrow{OB} + (\frac{l}{2} - \frac{n}{2})\overrightarrow{OC} = \overrightarrow{0}
\]
Since the vertices of the triangle $A$, $B$, and $C$ are not collinear, the position vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are not coplanar. Therefore, they are linearly independent vectors. This linear combination can only equal 0 if each of the coefficients separately equals zero:

\[
k - \frac{n}{2} = 0, \quad \frac{l}{2} - m = 0, \quad \frac{1}{2} - \frac{n}{2} = 0
\]

Since $k - \frac{n}{2} = 0$, $n = 2k$

Since $\frac{1}{2} - \frac{n}{2} = 0$, $l = n = 2k$

Now $k + l = 1$, so $k + 2k = 1$, or $k = \frac{1}{3}$

Then $l = \frac{2}{3}$

Then $\overrightarrow{OG} = \frac{1}{3}\overrightarrow{OA} + \frac{2}{3}\overrightarrow{OD}$, and $G$ divides $AD$ in the ratio 2:1.

Similarly, $\frac{1}{2} - m = 0$, $l = 2m$

and $\frac{1}{2} - \frac{n}{2} = 0$, $l = n = 2m$

Since $m + n = 1$, then $m = \frac{1}{3}$, $n = \frac{2}{3}$

Then $\overrightarrow{OG} = \frac{1}{3}\overrightarrow{OB} + \frac{2}{3}\overrightarrow{OE}$, and $G$ divides $BE$ in the ratio 2:1.

If you repeat this work using $\overrightarrow{AD}$, for instance, and the third median $\overrightarrow{CF}$, the result is the same. So the point of intersection $G$ divides each of the medians in the ratio 2:1. $G$ is called the **centroid** of the triangle.

Another type of problem asks for a proof that two line segments are perpendicular. Such problems are handled by showing that the dot product of the corresponding vectors is zero.

---

**EXAMPLE 4**

Prove that an angle inscribed in a semicircle is a right angle.

**Solution**

Let $O$ be the centre of a circle with diameter $AB$. Draw angle $\angle ACB$ in the semicircle. This angle is the angle between the vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$. If the dot product of the two vectors is zero, then $\angle C$ is a right angle.

\[
\overrightarrow{CA} \cdot \overrightarrow{CB} = (\overrightarrow{OA} - \overrightarrow{OC}) \cdot (\overrightarrow{OB} - \overrightarrow{OC})
\]

$OA$ and $OB$ are both radii and $\overrightarrow{OB} = -\overrightarrow{OA}$.

Then

\[
\overrightarrow{CA} \cdot \overrightarrow{CB} = (\overrightarrow{OA} - \overrightarrow{OC}) \cdot (-\overrightarrow{OA} - \overrightarrow{OC})
\]

\[
= -\overrightarrow{OA} \cdot \overrightarrow{OA} - \overrightarrow{OA} \cdot \overrightarrow{OC} + \overrightarrow{OC} \cdot \overrightarrow{OA} + \overrightarrow{OC} \cdot \overrightarrow{OC}
\]

\[
= -|\overrightarrow{OA}|^2 + |\overrightarrow{OC}|^2
\]

\[
= 0, \text{ since } |\overrightarrow{OA}| \text{ and } |\overrightarrow{OC}| \text{ are radii.}
\]

Therefore, $\angle ACB$ is a right angle.
EXAMPLE 5
If the diagonals of a parallelogram are perpendicular, prove that the parallelogram is a rhombus.

Solution
Draw and label a diagram.
Let \(ABCD\) be a parallelogram. Then opposite sides are equal vectors; for instance, \(\overrightarrow{AB} = \overrightarrow{DC}\). The diagonals are perpendicular, so

\[
\overrightarrow{AC} \cdot \overrightarrow{BD} = 0
\]

\[
(\overrightarrow{AB} + \overrightarrow{BC}) \cdot (\overrightarrow{BC} + \overrightarrow{CD}) = 0
\]

\[
(-\overrightarrow{CD} + \overrightarrow{BC}) \cdot (\overrightarrow{BC} + \overrightarrow{CD}) = 0
\]

\[
-\overrightarrow{CD} \cdot \overrightarrow{BC} - \overrightarrow{CD} \cdot \overrightarrow{CD} + \overrightarrow{BC} \cdot \overrightarrow{BC} + \overrightarrow{BC} \cdot \overrightarrow{CD} = 0
\]

\[
-|\overrightarrow{CD}|^2 + |\overrightarrow{BC}|^2 = 0
\]

Therefore, \(|\overrightarrow{BC}| = |\overrightarrow{CD}|\), so adjacent sides are equal and the figure must be a rhombus.

Exercise 6.4

Part A

Communication

1. Prove that the line joining the midpoints of two sides of a triangle is parallel to the third side and half its length
   a. using point-to-point vectors
   b. using position vectors

Knowledge/Understanding

2. If side BC of \(\triangle ABC\) is trisected by points \(P\) and \(Q\), show that \(\overrightarrow{AB} + \overrightarrow{AC} = \overrightarrow{AP} + \overrightarrow{AQ}\)
   a. using point-to-point vectors
   b. using position vectors

3. If \(D, E,\) and \(F\) are the midpoints of the sides of the triangle \(ABC\), prove that \(\overrightarrow{OD} + \overrightarrow{OE} + \overrightarrow{OF} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\).

Part B

4. Prove that if the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.
5. If $G$ is the centroid of $\triangle ABC$ and $AD$ is one of its medians,
   a. in what ratio does $D$ divide $BC$?
   b. in what ratio does $G$ divide $AD$?
   c. Prove that $\overrightarrow{OG} = \frac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC})$.

6. If $G$ is the centroid of $\triangle ABC$, prove that $\overrightarrow{AG} + \overrightarrow{BG} + \overrightarrow{CG} = \overrightarrow{0}$.

7. Prove that the diagonals of a parallelogram bisect each other. Use the type of proof shown in Example 3.

8. If a line through the centre of a circle is perpendicular to a chord, prove that it intersects the chord at its midpoint.

9. Show that the midpoint of the hypotenuse of a right-angled triangle is equidistant from the vertices.

10. Prove that the sum of the squares of the diagonals of any parallelogram is equal to the sum of the squares of the four sides.

11. In the trapezoid $ABCD$, $\overrightarrow{AB} = n\overrightarrow{DC}$. If the diagonals $BD$ and $AC$ meet at $K$, show that $\overrightarrow{AK} = \frac{n}{n+1}\overrightarrow{AD} + \frac{1}{n+1}\overrightarrow{AB}$.

12. $\triangle ABC$ is inscribed in a circle with centre $X$. Define a point $P$ by its position vector $\overrightarrow{XP} = \overrightarrow{XA} + \overrightarrow{XB} + \overrightarrow{XC}$.
   a. Show that $\overrightarrow{CP} = \overrightarrow{XA} + \overrightarrow{XB}$.
   b. Show that $\overrightarrow{CP} \perp \overrightarrow{AB}$, $\overrightarrow{BP} \perp \overrightarrow{AC}$, and $\overrightarrow{AP} \perp \overrightarrow{BC}$.
   c. Explain why the results of part b prove that the three altitudes of a triangle intersect at a common point. ($P$ is known as the orthocentre of the triangle.)

13. Let $ABCD$ be a rectangle. Prove that
   a. $\overrightarrow{OA} \cdot \overrightarrow{OC} = \overrightarrow{OB} \cdot \overrightarrow{OD}$
   b. $|\overrightarrow{OA}|^2 + |\overrightarrow{OC}|^2 = |\overrightarrow{OB}|^2 + |\overrightarrow{OD}|^2$

Part C

14. A regular hexagon $ABCDEF$ has two of its diagonals, $AC$ and $BE$, meeting at the point $K$. Determine the ratios in which $K$ divides $AC$ and $BE$. 
15. In a triangle $ABC$, the point $E$ is selected on $BC$ so that $BE:EC = 1:2$. The point $F$ divides $AC$ in the ratio 2:3. The two line segments $BF$ and $AE$ intersect at $D$.

a. Find the ratios in which $D$ divides $AE$ and $BF$.

b. Determine the ratio of the area of the quadrilateral $CEDF$ to the area of the triangle $ABC$.

16. In the parallelogram $ABCD$, $DC$ is extended to $E$ so that $DE:EC = 3:-2$. The line $AE$ meets $BC$ at $F$. Determine the ratios in which $F$ divides $BC$ and $F$ divides $AE$.

17. In the quadrilateral $APBQ$, $|AP| = |AQ|$ and $|BP| = |BQ|$.

a. Prove that $AB$ bisects $PQ$.

b. Prove that $AB$ is perpendicular to $PQ$.

18. Given the tetrahedron $MNPQ$ with $MN \perp PQ$ and $MP \perp NQ$, prove $MQ \perp NP$. 

Thinking/Inquiry/Problem Solving
The fundamental concept in this chapter is that of linear independence. If the linear combination \( au + bv + cw + \ldots \) is equal to 0 only if the coefficients \( a, b, c, \ldots \) are each zero, then the set of vectors \( u, v, w, \ldots \) is linearly independent.

You should understand what the implications are when a set of vectors is found to be linearly independent or not.

The question of linear independence is usually approached indirectly by asking if a set of vectors is linearly dependent. That consists of trying to express one of the vectors in the set in terms of the others. If the vectors are not linearly dependent, they must be linearly independent.

Two non-zero vectors \( \vec{u} \) and \( \vec{v} \) are linearly dependent
• if they are collinear
• if \( \vec{u} = k\vec{v} \) with \( k \neq 0 \), or
• if the cross product \( \vec{u} \times \vec{v} = \vec{0} \).

Three non-collinear vectors \( \vec{u}, \vec{v}, \) and \( \vec{w} \) are linearly dependent
• if they are coplanar
• if \( \vec{u} = a\vec{v} + b\vec{w} \) where \( a \) and \( b \) are not both zero, or
• if the triple scalar product \( \vec{u} \cdot \vec{v} \times \vec{w} = 0 \).

Three vectors in a two-dimensional plane and four vectors in three-dimensional space are always linearly dependent.

The division of a line segment is also connected to the linear independence of vectors. In the linear combination \( \overrightarrow{OP} = m\overrightarrow{OA} + n\overrightarrow{OB} \), the three points \( A, B, \) and \( P \) are collinear and their corresponding position vectors are coplanar only if the coefficients \( m + n = 1 \).

You should be able to express the division of a line segment \( AB \) by a point \( P \) in three equivalent ways and to convert readily from one to the other.

\[
P \text{ divides } AB \text{ in the ratio } a:b, \quad \frac{AP}{PB} = \frac{a}{b}, \quad \text{and} \quad \overrightarrow{OP} = \frac{b}{a+b} \overrightarrow{OA} + \frac{a}{a+b} \overrightarrow{OB}.
\]

This is not difficult, if you pay attention to the form of the equations and the positions of the letters representing the individual points.
The concept of linear independence and the properties of division points both play a role in vector proofs of geometrical propositions. Two approaches have been illustrated:

1. using point-to-point vectors that lie in the plane of the figure
2. using position vectors from some origin to points in the figure

It should be possible to prove a proposition using either approach. However, one approach may be more difficult than the other, and unfortunately, there is no way to predict this. If you can make no progress using one method, try another.

To learn to carry out proofs successfully, there is no substitute for doing many problems. Start with the simpler proofs. Follow the examples. Expect to work through and write out the logic of a proof several times until you get it right. Persevere.